Objective:

1. Description of systems using linear constant coefficient difference equations.
2. Representation of discrete-time signals and systems in the frequency domain.

Introduction:

- An important class of LTI systems consists of those systems for which the input \(x[n]\) and the output \(y[n]\) satisfy an \(N\)th-order linear constant-coefficient difference equation of the form

\[
\sum_{k=0}^{N} a_k y[n - k] = \sum_{m=0}^{M} b_m x[n - m].
\]

Solving these difference equations indicate finding the response of the LTI systems for different inputs.

- The Fourier representation of signals plays an extremely important role in both continuous-time and discrete-time signal processing. It provides a method for mapping signals into another "domain" in which to manipulate them. The Fourier representation is useful particularly in the form of a property that the convolution operation is mapped to multiplication. In addition, the Fourier transform provides a different way to interpret signals and systems. We will develop the discrete-time Fourier transform (i.e., a Fourier transform for discrete-time signals).

Description:

**Linear Constant Coefficient Difference Equations (LCCDE)**

The convolution sum expresses the output of a linear shift-invariant system in terms of a linear combination of the input values \(x(n)\). For example, a system that has a unit sample response \(h(n) = a^n u(n)\) is described by the equation

\[
y(n) = \sum_{k=0}^{\infty} a^k x(n - k)
\]

In some cases it may be possible to more efficiently express the output in terms of past values of the output in addition to the current and past values of the input. The previous system, for example, may be described more concisely as follows:

\[
y(n) = a y(n - 1) + x(n)
\]

The above equation is a special case of what is known as a linear constant coefficient difference equation, or LCCDE. The general form of a LCCDE is

\[
y(n) = \sum_{k=0}^{q} b_k x(n - k) - \sum_{k=1}^{p} a_k y(n - k)
\]

where the coefficients \(a_k\) and \(b_k\) are constants that define the system. If the difference equation has one or more terms \(a_k\) that are nonzero, the difference equation is said to be
recursive. On the other hand, if all of the coefficients \( a_k \) are equal to zero, the difference equation is said to be non-recursive. Thus, the equation
\[
y(n) = a y(n-1) + x(n)
\]
is an example of a first-order recursive difference equation, whereas
\[
y(n) = \sum_{k=0}^{\infty} a^k x(n-k)
\]
is an infinite-order non-recursive difference equation.

Difference equations provide a method for computing the response of a system, \( y(n) \), to an arbitrary input \( x(n) \). Before these equations may be solved, however, it is necessary to specify a set of initial conditions. For example, with an input \( x(n) \) that begins at time \( n = 0 \), the solution to general form of LCCDE at time \( n = 0 \) depends on the values of \( y(-1),..., y(-p) \). Therefore, these initial conditions must be specified before the solution for \( n \geq 0 \) may be found. When these initial conditions are zero, the system is said to be in initial rest.

For an LSI system that is described by a difference equation, the unit sample response, \( h(n) \), is found by solving the difference equation for \( x(n) = \delta(n) \) assuming initial rest. For a non-recursive system, \( a_k = 0 \), the difference equation becomes
\[
y(n) = \sum_{k=0}^{q} b_k x(n-k)
\]
and the output is simply a weighted sum of the current and past input values. As a result, the unit sample response is simply
\[
h(n) = \sum_{k=0}^{q} b_k \delta(n-k)
\]
Thus, \( h(n) \) is finite in length and the system is referred to as a finite-length impulse response (FIR) system. However, if \( a_k \neq 0 \), the unit sample response is, in general, infinite in length and the system is referred to as an infinite-length impulse response (IIR) system. For example, if
\[
y(n) = a y(n-1) + x(n)
\]
the unit sample response is \( h(n) = a^n \delta(n) \).

There are several different methods that one may use to solve LCCDEs for a general input \( x(n) \). The direct approach involves finding the homogeneous and particular solution and then writing the total output for given initial conditions. Another approach called as Indirect approach is to use z-transforms.

**Direct Approach for solving LCCDE**

Given an LCCDE, the general solution is a sum of two parts,
\[
y(n) = y_h(n) + y_p(n)
\]
where \( y_h(n) \) is known as the **homogeneous solution** and \( y_p(n) \) is the **particular solution**. The homogeneous solution is the response of the system to the initial conditions, assuming that the input \( x(n) = 0 \). The particular solution is the response of the system to the input \( x(n) \), assuming zero initial conditions.

The homogeneous solution is found by solving the homogeneous difference equation
\[ y(n) + \sum_{k=1}^{p} a_k y(n-k) = 0 \]

The solution to above equation may be found by assuming a solution of the form

\[ y_h(n) = z^n \]

Substituting this, we obtain the polynomial equation

\[ z^n + \sum_{k=1}^{p} a_k z^{n-k} = 0 \]

or

\[ z^{n-p} \{ z^p + a_1 z^{p-1} + a_2 z^{p-2} + \cdots + a_{p-1} z + a_p \} = 0 \]

The polynomial in braces is called the **characteristic polynomial**. Because it is of degree \( p \), it will have \( p \) roots, which may be either real or complex. If the coefficients \( a_k \) are real-valued, these roots will occur in complex conjugate pairs (i.e., for each complex root \( z \), there will be another that is equal to \( z^* \)). If the \( p \) roots \( z_i \) are distinct, \( z_i \neq z_k \) for \( k \neq i \), the general solution to the homogeneous difference equation is

\[ y_h(n) = \sum_{k=1}^{p} A_k z_i^n \]

where the constants \( A_k \) are chosen to satisfy the initial conditions. For repeated roots, the solution must be modified as follows. If \( z_i \) is a root of multiplicity \( m \) with the remaining \( p - m \) roots distinct, the homogeneous solution becomes

\[ y_h(n) = (A_1 + A_2 n + \cdots + A_m n^{m-1}) z_i^n + \sum_{k=m+1}^{p} A_k z_i^n \]

For the particular solution, it is necessary to find the sequence \( y_p(n) \) that satisfies the difference equation for the given \( x(n) \). For many of the typical inputs that we are interested in, the solution will have the same form as the input. Table 2.1 lists the particular solution for some commonly encountered inputs. For example, if \( x(n) = a^n u(n) \), the particular solution will be of the form

\[ y_p(n) = C a^n u(n) \]

provided \( a \) is not a root of the characteristic equation. The constant \( C \) is found by substituting the solution into the difference equation. Note that for \( x(n) = C \delta(n) \) the particular solution is zero. Because \( x(n) = 0 \) for \( n > 0 \), the unit sample only affects the initial condition of \( y(n) \).

<table>
<thead>
<tr>
<th>Term in ( x(n) )</th>
<th>Particular Solution</th>
</tr>
</thead>
<tbody>
<tr>
<td>( C )</td>
<td>( C )</td>
</tr>
<tr>
<td>( C n )</td>
<td>( C_1 n + C_2 )</td>
</tr>
<tr>
<td>( C a^n )</td>
<td>( C_1 a^n )</td>
</tr>
<tr>
<td>( C \cos(n \omega_0) )</td>
<td>( C_1 \cos(n \omega_0) + C_2 \sin(n \omega_0) )</td>
</tr>
<tr>
<td>( C \sin(n \omega_0) )</td>
<td>( C_1 \cos(n \omega_0) + C_2 \sin(n \omega_0) )</td>
</tr>
<tr>
<td>( C a^n \cos(n \omega_0) )</td>
<td>( C_1 a^n \cos(n \omega_0) + C_2 a^n \sin(n \omega_0) )</td>
</tr>
<tr>
<td>( C \delta(n) )</td>
<td>None</td>
</tr>
</tbody>
</table>

Table 2.1 The Particular Solution to an LCCDE for Several Different Inputs
Frequency Response

Eigen functions of linear shift-invariant systems are sequences that, when input to the system, pass through with only a change in (complex) amplitude. That is to say, if the input is \( x(n) \), the output is \( y(n) = \lambda x(n) \), where \( \lambda \), the eigen value, generally depends on the input \( x(n) \).

Signals of the form

\[
x(n) = e^{j\omega n} \quad -\infty < n < \infty
\]

where \( \omega \) is a constant, are eigen functions of LSI systems. This may be shown from the convolution sum:

\[
y(n) = h(n) \ast x(n) = \sum_{k=-\infty}^{\infty} h(k)x(n-k)
\]

\[
= \sum_{k=-\infty}^{\infty} h(k)e^{j\omega(n-k)} = e^{j\omega} \sum_{k=-\infty}^{\infty} h(k)e^{-jk\omega}
\]

Thus, the eigen value, which we denote by \( H(e^{j\omega}) \), is

\[
H(e^{j\omega}) = \sum_{k=-\infty}^{\infty} h(k)e^{-jk\omega}
\]

Note that \( H(e^{j\omega}) \) is, in general, complex-valued and depends on the frequency \( \omega \) of the complex exponential. Thus, it may be written in terms of its real and imaginary parts.

\[
H(e^{j\omega}) = H_R(e^{j\omega}) + jH_I(e^{j\omega})
\]

or in terms of its magnitude and phase,

\[
H(e^{j\omega}) = |H(e^{j\omega})|e^{j\phi(h(\omega))}
\]

where

\[
|H(e^{j\omega})|^2 = H(e^{j\omega})H^*(e^{j\omega}) = H_R^2(e^{j\omega}) + H_I^2(e^{j\omega})
\]

and

\[
\phi_\omega(\omega) = \tan^{-1} \left( \frac{H_I(e^{j\omega})}{H_R(e^{j\omega})} \right)
\]

Discrete Time Fourier Transform (DTFT)

Many sequences can be represented by a Fourier integral of the form

\[
x[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})e^{j\omega n} d\omega,
\]

.......................(1)

where

\[
X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x[n]e^{-j\omega n}.
\]

.......................(2)
Equations (1) and (2) together form a Fourier representation for the sequence. Equation (1), the inverse Fourier transform, is a synthesis formula. That is, it represents \( x[n] \) as a superposition of infinitesimally small complex sinusoids of the form

\[
\frac{1}{2\pi} X(e^{j\omega})e^{j\omega n} d\omega,
\]

with \( \omega \) ranging over an interval of length \( 2\pi \) and with \( X(e^{j\omega}) \) determining the relative amount of each complex sinusoidal component. Although, in writing Eq. (1), we have chosen the range of values for \( \omega \) between \( -\pi \) and \( +\pi \), any interval of length \( 2\pi \) can be used. Eq. (2), the Fourier transform, is an expression for computing \( X(e^{j\omega}) \) from the sequence \( x[n] \), i.e., for analyzing the sequence \( x[n] \) to determine how much of each frequency component is required to synthesize \( x[n] \) using Eq. (1). In general, the Fourier transform is a complex-valued function of \( \omega \). As with the frequency response, we may either express \( X(e^{j\omega}) \) in rectangular form as

\[
X(e^{j\omega}) = X_R(e^{j\omega}) + jX_I(e^{j\omega})
\]

or in polar form as

\[
X(e^{j\omega}) = |X(e^{j\omega})| e^{j\angle X(e^{j\omega})},
\]

with \( |X(e^{j\omega})| \) representing the magnitude and \( \angle X(e^{j\omega}) \) the phase.

A sufficient condition for convergence can be found as follows:

\[
|X(e^{j\omega})| = \left| \sum_{n=-\infty}^{\infty} x[n] e^{-j\omega n} \right|
\leq \sum_{n=-\infty}^{\infty} |x[n]| |e^{-j\omega n}|
\leq \sum_{n=-\infty}^{\infty} |x[n]| < \infty.
\]

Thus, if \( x[n] \) is absolutely summable, then \( X(e^{j\omega}) \) exists

Using this concept, the impulse response can be obtained from the frequency response by applying the inverse Fourier transform integral; i.e.,

\[
h[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} H(e^{j\omega})e^{j\omega n} d\omega,
\]

where \( H(e^{j\omega}) \) is the frequency response of the system.

**Properties of DTFT**

The Table 2.2 summarizes the properties or theorems of Fourier Transforms
Table 2.2 Properties of DTFT

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n] + y[n]$</td>
<td>$aX(e^{j\omega}) + bY(e^{j\omega})$</td>
</tr>
<tr>
<td>$x[-n]$</td>
<td>$X(-e^{-j\omega}) = X^*(e^{j\omega})$ if $x[n]$ real.</td>
</tr>
<tr>
<td>$nx[n]$</td>
<td>$\frac{dX(e^{j\omega})}{d\omega}$</td>
</tr>
<tr>
<td>$x[n] \leftrightarrow y[n]$</td>
<td>$X(e^{j\omega})Y(e^{j\omega})$</td>
</tr>
<tr>
<td>$x[n]y[n]$</td>
<td>$\frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y(e^{j(\omega-\theta)})d\theta$</td>
</tr>
</tbody>
</table>

Parseval’s theorem:

8. $\sum_{n=-\infty}^{\infty} |x[n]|^2 = \frac{1}{2\pi} \int_{-\pi}^{\pi} |X(e^{j\omega})|^2d\omega$

9. $\sum_{n=-\infty}^{\infty} x[n]^*y[n] = \frac{1}{2\pi} \int_{-\pi}^{\pi} X(e^{j\omega})Y^*(e^{j\omega})d\omega$

The symmetry properties of the Fourier transform are summarized in Table 2.3

Table 2.3 Symmetry Properties of Fourier Transform

<table>
<thead>
<tr>
<th>Sequence</th>
<th>Fourier Transform</th>
</tr>
</thead>
<tbody>
<tr>
<td>$x[n]$</td>
<td>$X(e^{j\omega})$</td>
</tr>
<tr>
<td>$x[-n]$</td>
<td>$X^*(e^{-j\omega})$</td>
</tr>
<tr>
<td>$\Re{x[n]}$</td>
<td>$X_R(e^{j\omega})$ (conjugate-symmetric part of $X(e^{j\omega})$)</td>
</tr>
<tr>
<td>$\Im{x[n]}$</td>
<td>$X_I(e^{j\omega})$ (conjugate-antisymmetric part of $X(e^{j\omega})$)</td>
</tr>
<tr>
<td>$x_R[n]$ (conjugate-symmetric part of $x[n]$)</td>
<td>$X_R(e^{j\omega}) = \Re{X(e^{j\omega})}$</td>
</tr>
<tr>
<td>$x_I[n]$ (conjugate-antisymmetric part of $x[n]$)</td>
<td>$X_I(e^{j\omega}) = \Im{X(e^{j\omega})}$</td>
</tr>
</tbody>
</table>

The following properties apply only when $x[n]$ is real:

7. Any real $x[n]$ | $X(e^{j\omega}) = X^*(e^{-j\omega})$ (Fourier transform is conjugate symmetric) |
8. Any real $x[n]$ | $X_R(e^{j\omega}) = X_R(e^{-j\omega})$ (real part is even) |
9. Any real $x[n]$ | $X_I(e^{j\omega}) = -X_I(e^{-j\omega})$ (imaginary part is odd) |
10. Any real $x[n]$ | $|X(e^{j\omega})| = |X(e^{-j\omega})|$ (magnitude is even) |
11. Any real $x[n]$ | $\angle X(e^{j\omega}) = -\angle X(e^{-j\omega})$ (phase is odd) |
12. $x_{e}[n]$ (even part of $x[n]$) | $X_R(e^{j\omega})$ |
13. $x_{o}[n]$ (odd part of $x[n]$) | $jX_I(e^{j\omega})$ |

Fourier transform is a sum, whereas the inverse transform is an integral with a periodic integrand. The Fourier transform of some basic sequences is summarized in Table 2.4
We present some applications of the DTFT in discrete-time signal analysis. These include finding the frequency response of an LSI system that is described by a difference equation, performing convolutions, solving difference equations that have zero initial conditions.

1) **LSI Systems and LCCDEs**

An important subclass of LSI systems contains those whose input, \( x(n) \), and output, \( y(n) \), are related by a linear constant coefficient difference equation (LCCDE):

\[
y(n) = - \sum_{k=1}^{p} a_k y(n - k) + \sum_{k=0}^{q} b_k x(n - k)
\]

The linearity and shift properties of the DTFT may be used to express this difference equation in the frequency domain as follows:

\[
Y(e^{j\omega}) = - \sum_{k=1}^{p} a_k Y(e^{j\omega}) e^{-j\omega k} + \sum_{k=0}^{q} b_k X(e^{j\omega}) e^{-j\omega k}
\]
\[ Y(e^{j\omega}) \left\{ 1 + \sum_{k=1}^{p} a_k e^{-j\omega k} \right\} = X(e^{j\omega}) \left\{ \sum_{k=0}^{q} b_k e^{-j\omega k} \right\} \]

Therefore, the frequency response of this system is

\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} = \frac{\sum_{k=0}^{q} b_k e^{-j\omega k}}{1 + \sum_{k=1}^{p} a_k e^{-j\omega k}} \]

2) Performing Convolutions

Because the DTFT maps convolution in the time domain into multiplication in the frequency domain, the DTFT provides an alternative to performing convolutions in the time domain.

3) Solving Difference Equations

The DTFT may be used to solve difference equations in the "frequency domain" provided that the initial conditions are zero. The procedure is simply to transform the difference equation into the frequency domain by taking the DTFT of each term in the equation, solving for the desired term, and finding the inverse DTFT.

**Illustrative Examples:**

**Problem 1:** Find the solution to the difference equation

\[ y(n) - 0.25y(n - 2) = x(n) \]

for \( x(n) = u(n) \) assuming initial conditions of \( y(-1) = 1 \) and \( y(-2) = 0 \).

**Solution:**

We begin by finding the particular solution. From Table 2.1 we see that for \( x(n) = u(n) \)

\[ y_p(n) = C_1 \]

Substituting this solution into the difference equation we find

\[ C_1 - 0.25C_1 = 1 \]

In order for this to hold, we must have

\[ C_1 = \frac{1}{1 - 0.25} = \frac{4}{3} \]

To find the homogeneous solution, we set \( y_h(n) = z^n \), which gives the characteristic polynomial

\[ z^2 - 0.25 = 0 \]

or

\[ (z + 0.5)(z - 0.5) = 0 \]

Therefore, the homogeneous solution has the form
Thus, the total solution is
\[ y(n) = \frac{4}{3} + A_1(0.5)^n + A_2(-0.5)^n \quad n \geq 0 \]
The constants \( A_1 \) and \( A_2 \) must now be found so that the total solution satisfies the given initial conditions, \( y(-1) = 1 \) and \( y(-2) = 0 \). Because the solution given in the above equation only applies for \( n \geq 0 \), we must derive an equivalent set of initial conditions for \( y(0) \) and \( y(1) \). Evaluating the LCCDE
\[ y(n) - 0.25y(n-2) = x(n) \]
at \( n = 0 \) and \( n = 1 \), we have
\[
\begin{align*}
y(0) - 0.25y(-2) &= x(0) = 1 \\
y(1) - 0.25y(-1) &= x(1) = 1
\end{align*}
\]
Substituting these derived initial conditions into the total solution obtained containing \( A_1 \) and \( A_2 \), we have
\[
\begin{align*}
y(0) &= \frac{4}{3} + A_1 + A_2 = 1 \\
y(1) &= \frac{4}{3} + \frac{1}{2}A_1 - \frac{1}{2}A_2 = 1
\end{align*}
\]
Solving for \( A_1 \) and \( A_2 \) we find
\[ A_1 = -\frac{1}{2} \quad A_2 = \frac{1}{6} \]
Thus, the solution is
\[ y(n) = \frac{4}{3} - (0.5)^{n+1} + \frac{1}{6}(-0.5)^n \quad n \geq 0 \]

**Problem 2:** Find the Fourier transform of
\[ x_1(n) = \alpha^n u(n) \quad |\alpha| < 1 \]

**Solution:**
Applying the DTFT
\[ X_1(e^{j\omega}) = \sum_{n=0}^{\infty} \alpha^n e^{-j\omega n} = \sum_{n=0}^{\infty} (\alpha e^{-j\omega})^n \]
Using the geometric series, this sum is
\[ X_1(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \]
provided \( |\alpha| < 1 \).

**Problem 3:** Consider the linear shift-invariant system characterized by the second-order linear constant coefficient difference equation
\[ y(n) = 1.3433y(n-1) - 0.9025y(n-2) + x(n) - 1.4142x(n-1) + x(n-2) \]
Find the frequency response.

**Solution:**
The frequency response may be found by applying the DTFT on both sides and taking the ratio of output spectrum to input spectrum
\[ H(e^{j\omega}) = \frac{Y(e^{j\omega})}{X(e^{j\omega})} \]
Therefore for the difference equation given
\[ H(e^{j\omega}) = \frac{1 - 1.4142e^{-j\omega} + e^{-2j\omega}}{1 - 1.3433e^{-j\omega} + 0.9025e^{-2j\omega}} \]

**Problem 4:** If the unit sample response of an LSI system is
\[ h(n) = \alpha^n u(n) \]
find the response of the system to the input \( x(n) = \beta^n u(n) \), where \(|\alpha| < 1, |\beta| < 1\), and \( \alpha \neq \beta \).

**Solution:**
Because the output of the system is the convolution of \( x(n) \) with \( h(n) \),
\[ y(n) = h(n) * x(n) \]
the DTFT of \( y(n) \) is
\[ Y(e^{j\omega}) = H(e^{j\omega})X(e^{j\omega}) = \frac{1}{1 - \alpha e^{-j\omega}} \frac{1}{1 - \beta e^{-j\omega}} \]
Therefore, all that is required is to find the inverse DTFT of \( Y(e^{j\omega}) \). This may be done easily by expanding \( Y(e^{j\omega}) \) as follows:
\[ Y(e^{j\omega}) = \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})} = \frac{A}{1 - \alpha e^{-j\omega}} + \frac{B}{1 - \beta e^{-j\omega}} \]
where \( A \) and \( B \) are constants that are to be determined. Expressing the right-hand side of this expansion over a common denominator,
\[ \frac{1}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})} = \frac{(A + B) - (A\beta + B\alpha) e^{-j\omega}}{(1 - \alpha e^{-j\omega})(1 - \beta e^{-j\omega})} \]
and equating coefficients, the constants \( A \) and \( B \) may be found by solving the pair of equations
\[ A + B = 1 \]
\[ A\beta + B\alpha = 0 \]
The result is
\[ A = \frac{\alpha}{\alpha - \beta} \quad B = -\frac{\beta}{\alpha - \beta} \]
Therefore,
\[ Y(e^{j\omega}) = \frac{\alpha/(\alpha - \beta)}{1 - \alpha e^{-j\omega}} - \frac{\beta/(\alpha - \beta)}{1 - \beta e^{-j\omega}} \]
and it follows that the inverse DTFT is
\[ y(n) = \left[ \frac{\alpha}{\alpha - \beta} \alpha^n - \frac{\beta}{\alpha - \beta} \beta^n \right] u(n) \]

**Problem 5:**
Solve the following LCCDE for \( y(n) \) for \( x(n) = \delta(n) \), assuming zero initial conditions.
\[ y(n) - 0.25y(n - 1) = x(n) - x(n - 2) \]

**Solution:**
We begin by taking the DTFT of each term in the difference equation:
\[ Y(e^{j\omega}) - 0.25e^{-j\omega}Y(e^{j\omega}) = X(e^{j\omega}) - e^{-2j\omega}X(e^{j\omega}) \]

Because the DTFT of \( x(n) = \delta(n) \) is \( X(e^{j\omega}) = 1 \),

\[ Y(e^{j\omega}) = \frac{1 - e^{-2j\omega}}{1 - 0.25e^{-j\omega}} = \frac{1}{1 - 0.25e^{-j\omega}} - \frac{e^{-2j\omega}}{1 - 0.25e^{-j\omega}} \]

Using the DTFT pair

\[ (0.25)^n u(n) \rightleftharpoons \frac{1}{1 - 0.25e^{-j\omega}} \]

the inverse DTFT of \( Y(e^{j\omega}) \) may be easily found using the linearity and shift properties,

\[ y(n) = (0.25)^n u(n) - (0.25)^{n-2} u(n - 2) \]

**Summary:**

To summarize, for a system for which the input and output satisfy a linear constant coefficient difference equation, the output for a given input is not uniquely specified. Auxiliary information or initial conditions are required. Linearity, time invariance, and causality of the system will depend on the auxiliary conditions. If an additional condition is that the system is initially at rest, then the system will be linear, time invariant, and causal.

**Assignment:**

**Problem 1:** Find the Fourier Transform of

\[ x_2(n) = -\alpha^n u(-n - 1) \quad |\alpha| > 1 \]

**Problem 2:** Suppose \( X(e^{j\omega}) \) consists of an impulse at frequency \( \omega = \omega_0 \). Find the Inverse DTFT

**Problem 3:** Find the difference equation representation of the system if its frequency response is

\[ H(e^{j\omega}) = \frac{0.5 + 0.5e^{-2j\omega}}{1 - 0.5e^{-j\omega} + 0.25e^{-2j\omega}} \]

**Problem 4:** The inverse of a system with unit sample response \( h(n) \) is a system that has a unit sample response \( g(n) \) such that

\[ h(n) * g(n) = \delta(n) \]

If \( H(e^{j\omega}) = 1 - 2e^{-j\omega} \), find \( g(n) \).

**Problem 5:** Show that if \( X(e^{j\omega}) \) is real and even, \( x(n) \) is real and even.

**Simulation:**

To Calculate Unit Impulse Response (Unit Sample), Unit Step Response of the given LTI system

\[ y(n) = (3/8)y(n-1) + (2/3)y(n-2) + x(n) + (1/4)x(n-1) \]

**Input:**

type the numerator vector [1,1/4]
type the denominator vector $[1, -3/8, -2/3]$

**Program:**

```matlab
% To calculate Unit Impulse response
clc;
clear all;
close all;
% Given system
% \( y(n) = \frac{3}{8}y(n-1) + \frac{2}{3}y(n-2) + x(n) + \frac{1}{4}x(n-1) \)
num = input ('type the numerator vector ');
den = input ('type the denominator vector ');
N = input ('type the desired length of the output sequence N ');
n = 0 : N-1;
h=impz(num,den);
disp('The impulse response of LTI system is'); disp(h(1:N));
stem(n,h(1:N))
xlabel ('time index n');
ylabel ('Amplitude ');
title ('Impulse Response of LTI system');

% To calculate Unit Step response
% Given system
% \( y(n) = \frac{3}{8}y(n-1) + \frac{2}{3}y(n-2) + x(n) + \frac{1}{4}x(n-1) \)
num = input ('type the numerator vector ');
den = input ('type the denominator vector ');
N = input ('type the desired length of the output sequence N ');
n = 0 : 1 : N-1;
s=stepz(num,den);
disp('The step response of LTI system is'); disp(s(1:N));
stem(n,s(1:N))
xlabel ('time index n');
ylabel ('s(n)');
title ('Step Response of LTI system');
```

![Impulse Response of LTI System](image1)

![Step Response of LTI System](image2)
% To calculate Sinusoidal response
% Given system
% \( y(n) = \frac{3}{8}y(n-1) + \frac{2}{3}y(n-2) + x(n) + \frac{1}{4}x(n-1) \)
num = input ('type the numerator vector ');
den = input ('type the denominator vector ');
n = 0 : 0.1 : 2*pi;
in = sin(n);
s = filter (num, den, in);
subplot(2,1,1);
stem(n,in);
xlabel ('time index n');
ylabel ('(n)');
title('Input sinusoidal sequence');
subplot(2,1,2);
stem(n,s);
xlabel ('time index n');
ylabel ('s(n)');
title('Sinusoidal Response of LTI system');

References: