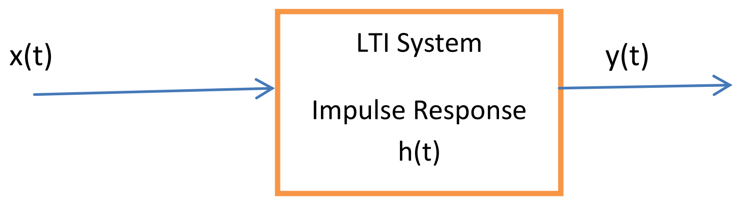
Module 29:Spectral characteristics of system response

Objective: The quality of a communication system deals with the delivery of message to the user, who is available after the receiver. The present module deals with the effect of an LTI system on the input random process. This in turn helps in the computation of S/N at the output of thr receiver in a communication system.

Module Description:

- **Linear System's response for a random Signal as input:**

➢ Let the input for an LTI systemof impulse response h(t), be a WSS process x(t), resulting an output y(t).



➢ The response of the system y(t)= x(t)*h(t)= $\int_{-\infty}^{\infty} h(\tau).x(t-\tau).d\tau$

❖ **Mean value of the Output Process**

➢ Expected Value of the output is $E[y(t)] = E\left[\int_{-\infty}^{\infty} h(\tau).x(t-\tau).d\tau\right] =$

$$= \int_{-\infty}^{\infty} h(\tau).E[x(t-\tau)].d\tau$$

➢ Since x(t) is WSS, $E[x(t-\tau)] = E[x(t)] = Constant m_x$.

➢ Then, $E[y(t)] = m_x.\int_{-\infty}^{\infty} h(\tau).d\tau$

➢ For an LTI system, Unit impulse response h(t) and its Transfer function H(ω) form a Fourier Transform Pair, i.e.

$$H(\omega) = \int_{-\infty}^{\infty} h(t).e^{-j\omega t}\ .dt$$

➢ Then, $H(0) = \int_{-\infty}^{\infty} h(t).dt$

➢ Hence, $E[y(t)] = m_x.H(0)$. Thus, the mean of the output process is independent of time and is a constant.

❖ **Autocorrelation Function of the Output Process**

➢ $R_{yy}(t_1,t_2) = E[y(t_1).y(t_2)] =$

$$E\left[\int_{-\infty}^{\infty} h(\tau_1).x(t_1-\tau_1).d\tau_1.\int_{-\infty}^{\infty} h(\tau_2).x(t_2-\tau_2).d\tau_2\right]$$

$=\int_{-\infty}^{\infty} h(\tau_1).h(\tau_2).E[x(t_1-\tau_1)x(t_2-\tau_2)].d\tau_1.d\tau_2$
$=\int_{-\infty}^{\infty} h(\tau_1).h(\tau_2).R_{xx}(t_2-\tau_2-t_1+\tau_1).d\tau_1.d\tau_2$
$=\int_{-\infty}^{\infty} h(\tau_1).h(\tau_2).R_{xx}(\tau-\tau_2+\tau_1).d\tau_1.d\tau_2$

where $\tau = t_2 - t_1$.

- Thus, the Autocorrelation function of the output process is independent of time and is dependent on "τ".
- Hence, y(t) is a stationary process.
- Thus, for an LTI system, if the input process is stationary, the resulting output process is also stationary.

❖ **Mean Squared Value of the Output Process**

- $R_{yy}(t_1,t_2) = \int_{-\infty}^{\infty} h(\tau_1).h(\tau_2).R_{xx}(\tau-\tau_2+\tau_1).d\tau_1.d\tau_2$
- Since, y(t) is stationary, the above equation can be written as

$$R_{yy}(\tau) = \int_{-\infty}^{\infty} h(\tau_1).h(\tau_2).R_{xx}(\tau-\tau_2+\tau_1).d\tau_1.d\tau_2$$

- The MS value of y(t) is $E[y^2(t)] = R_{yy}(0) = \int_{-\infty}^{\infty} h(\tau_1).h(\tau_2).R_{xx}(\tau_1-\tau_2).d\tau_1.d\tau_2$

❖ **Cross correlation Function between the input process and output process**

- $R_{xy}(\tau) = E[x(t).y(t+\tau)] = E[x(t).\int_{-\infty}^{\infty} h(\alpha).x(t+\tau-\alpha).d\alpha]$
  $= \int_{-\infty}^{\infty} E[x(t).x(t+\tau-\alpha)].h(\alpha).d\alpha$
  $=\int_{-\infty}^{\infty} R_{xx}(\tau-\alpha).h(\alpha).d\alpha = R_{xx}(\tau) * h(\tau) ----1.$
- In the above eq.1, replace $\tau by - \tau$ i.e $R_{xy}(-\tau) = R_{yx}(\tau) = R_{xx}(-\tau) * h(-\tau)$.
  Since, Autocorrelation function is an even function of $\tau, R_{yx}(\tau) = R_{xx}(\tau) * h(-\tau) --2.$
- In the above eq.2, replace second 'x' in the prefix by 'y' i.e.

$$R_{yy}(\tau) = R_{xy}(\tau) * h(-\tau) — 3$$

- In the above eq.3, replace $\tau by - \tau$, i.e.$R_{yy}(-\tau) = R_{xy}(-\tau) * h(\tau)$.
- It same as $R_{yy}(\tau)$, since Autocorrelation is having even symmetry.
- Since, $R_{xy}(-\tau) = R_{yx}(\tau), R_{yy}(\tau) = R_{yx}(\tau) * h(\tau)$----4.

❖ **Relation between the PSDs of the input process and output processof an LTI Systems**

- Take Fourier Transform on both sides for eq.2
- $S_{yx}(\omega) = S_{xx}(\omega)H^*(\omega)$ - - - 5
- Take Fourier Transform on both sides for eq.4
- $S_{yy}(\omega) = S_{yx}(\omega)H(\omega)$ - - - 6
- **Substituting** $S_{yx}(\omega)$ from eq.5 in eq.6
- $S_{yy}(\omega) = S_{xx}(\omega)H^*(\omega)H(\omega) = S_{xx}(\omega).|H(\omega)|^2$

- **The power at the output of the LTI system is the area enclosed by the output PSD i.e.**
- Output Power=$\frac{\mathbf{1}}{\mathbf{2\pi}}\int_{-\infty}^{\infty} S_{yy}(\omega).d\omega$

<u>Illustrative Examples:</u>

1. The input to an LTI system with Impulse response h(t)=δ(t)+t.exp(-at).U(t) is a WSS process with mean of 3.Find the mean of the output of the system.

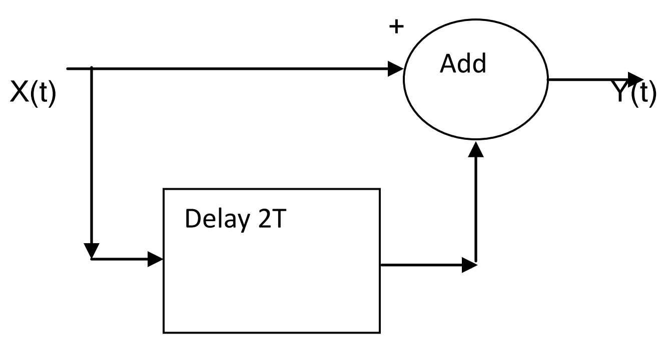
Soln.:

$$E[y(t)] = m_x.H(0)$$

$$H(\omega) = F[h(t)] = 1 + \frac{1}{(j\omega + a)^2} \text{ and } H(0) = \frac{a^2+1}{a^2}$$

Hence, $E[y(t)] = \frac{3(a^2+1)}{a^2}$

2. A WSS process X(t) is applied to the following System:



If the input PSD is 'K 'Watts/Hz, find the output PSD

Soln.:

The system is represented as $Y(t) = X(t) + X(t-2T)$

The Transfer function of the above system is $H(\omega) = 1 + e^{-j\omega\, 2T}$ and the corresponding

$$|H(\omega)|^2 = 2 + 2.Cos2\omega T$$

The output PSD is $S_{yy}(\omega) = S_{xx}(\omega).|H(\omega)|^2 = 2K(1 + Cos2\omega T)$ W/Hz.

3.A noise process with zero mean and of PSD "K" is applied to an R-L LPF.

Find the Mean Square value of the output Process.

Soln.: The Transfer Function of RL HPF is $H(f) = \frac{R}{R+j\omega L.}$ and the corresponding

$$|H(\omega)|^2 = \frac{R^2}{(R^2 + (\omega L)^2)}$$

The PSD at the output of the filter is $S_{yy}(\omega) = \frac{KR^2}{(R^2+(\omega L)^2)}$

The MS value of the output process is the area under output PSD**,** and is equal to $\boldsymbol{\frac{KR}{2L}}$

Exercise Problems:

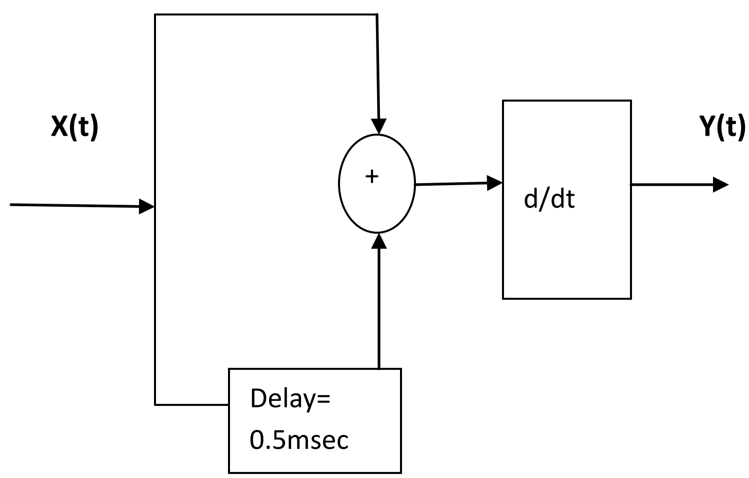
1.A white Gaussian noise process of zero mean and of PSD ŋ/2 is applied as the input of RL HPF. Find the Auto correlation and the PSD of the output of the filter.Find the Mean and variance of the output.

2.Two identical systems with impulse response $t.e^{-kt}$ for t>0; and 0 for t<0are cascaded. The output of the second system is Y (t).If the mean of the input of the First system is 2, find the mean of Y(t).

3.A WSS process X(t) is having an Autocorrelation function $R_{xx}(\tau)$ and the PSD $S_{xx}(\omega)$. Let

$Y(t) = \frac{d}{dt}X(t)$ Then, verify that (i) $R_{yy}(\tau) = -\frac{d^2}{d\tau^2}R_{xx}(\tau)$ (ii) $R_{xy}(\tau) = \frac{d}{d\tau}R_{xx}(\tau)$

4.X(t) is a stationary process with the PSD $S_x(f) > 0$. The process is passed through a system shown below:



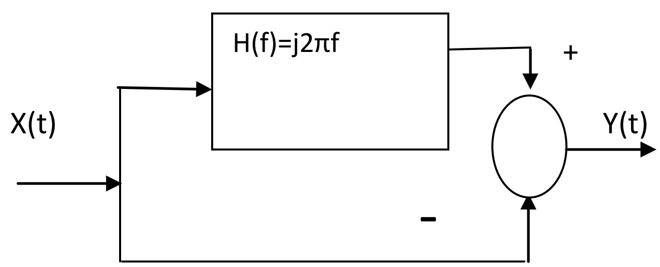
Let $S_y(f)$ be the PSD of Y(t). Then, which of the following is correct?

a)$S_y(f) > 0$ for all 'f' b) $S_y(f) > 0$ for all |f|>1KHz

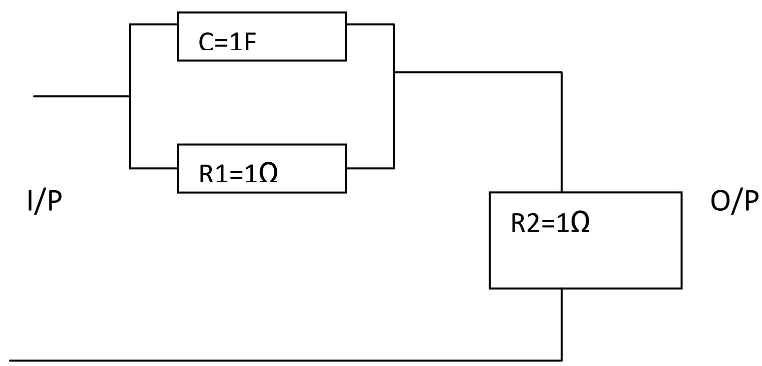
c)$S_y(f) = 0$ for all f=nf$_0$,f$_0$ = 2KHz, n is any integer

d)$S_y(f) = 0$ for all f=(2n+1)f$_0$,f$_0$ = 1KHz, n is any integer

5.X(t) is a real stationary process with Autocorrelation function $R_{xx}(\tau) = e^{-\pi\tau^2}$. The process is passed through the system shown below. Find the PSD of the output.



6.



A WSS Process X(t) with Auto correlation function .exp(-4|τ|) is applied to the above network, to give an output of Y(t).Find a)$S_{XY}(w)$ b)$S_{YY}(w)$.

7.A random process X(t) with PSD "K.[(w^2+9)/(w^2+16)] applied to a system with transfer function H(w) results in an output process Y(t) with a PSFD of "k". Find H(w).

8.White Noise with PSD of N_0 over(-∞,∞) is passed through a BPF With the following Frequency Response. Find the output noise Power .