Module 6
Introduction to Fourier series

Objective: To understand Fourier series representation of Periodic signals.

Introduction to Fourier Series:
A signal is said to be a continuous time signal if it is available at all instants of time. A real time naturally available signal is in the form of time domain. However, the analysis of a signal is far more convenient in the frequency domain. These are three important classes of transformation methods available for continuous-time systems. They are:

1. Fourier series
2. Fourier Transform
3. Laplace Transform

Out of these three methods, the Fourier series is applicable only to periodic signals, i.e. signals which repeat periodically over \(-\infty < t < \infty\). Not all periodic signals can be represented by Fourier series.

Description:
Basic idea of Fourier series is to project periodic signals onto a set of basis functions. The basis functions are orthogonal and span the space of periodic functions. Any periodic signal can be written as a weighted sum of these basis functions.

Representation of Fourier Series
The representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions is called Fourier Series. The Fourier analysis is also sometimes called the harmonic analysis. Fourier series is applicable only for periodic signals. It cannot be applied to non-periodic signals. A periodic signal is one which repeats itself at regular intervals of time, i.e. periodically over \(-\infty \) to \(\infty\). Three important classes of Fourier series methods are available. They are:

1. Trigonometric form
2. Cosine form
3. Exponential form

In the representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions, if the orthogonal functions are exponential functions then it is called exponential fourier series.

Similarly, in the representation of signals over a certain interval of time in terms of the linear combination of orthogonal functions, if the orthogonal functions are trigonometric functions, then it is called trigonometric Fourier series.

Existence of Fourier series
For the Fourier series to exist for a periodic signal, it must satisfy certain conditions.

The conditions under which a periodic signal can be represented by a Fourier series are known as Dirichlet’s conditions named after the mathematician Dirichlet who first found them. They are as follows:

In each period,

1. The function \(x(t)\) must be a single valued function.
2. The function \(x(t)\) has only a finite number of maxima and minima.

Example: This function \(x(t)\) violates Condition 2
3. The function $x(t)$ has a finite number of discontinuities.
   Example: This function violates condition 3

4. The function $x(t)$ is absolutely integrable over one period, that is $\int_0^T |x(t)| \, dt < \infty$.
   These are the sufficient but not necessary conditions for the existence of the Fourier series of a periodic function $x(t)$. Condition 4 is known as the weak Dirichlet condition. If a function satisfies the weak Dirichlet condition, the existence of Fourier series is guaranteed, but the series may not converge at every point. Conditions 2 and 3 are known as strong Dirichlet conditions. If these are satisfied, the convergence is also guaranteed.

**Trigonometric form of Fourier series:**

A sinusoidal signal, $x(t)=A \sin \omega_0 t$ is a periodic signal with period $T=2\pi/\omega_0$. Also, the sum of two sinusoids is periodic provided that their frequencies are integral multiples of a fundamental frequency $\omega_0$. We can show that a signal $x(t)$, a sum of sine and cosine functions whose frequencies are integral multiples of $\omega_0$, is a periodic signal.

Let the signal $x(t)$ be

$x(t)= a_0+a_1 \cos \omega_0 t+a_2 \cos 2\omega_0 t+ \ldots + a_k \cos k\omega_0 t + b_1 \sin \omega_0 t+b_2 \sin 2\omega_0 t+ \ldots + b_k \sin k\omega_0 t$

i.e.

$x(t)= a_0+\sum_{n=1}^{k}[a_n \cos \omega_0 nt + b_n \sin \omega_0 nt]$

where $a_0$, $a_1$, $a_2$, $\ldots$, $a_k$ and $b_0$, $b_1$, $b_2$, $\ldots$, $b_k$ are constants, and $\omega_0$ is the fundamental frequency.

For the signal $x(t)$ to be periodic, it must satisfy the condition $x(t) = x(t+T)$ for all $t$.

i.e.

$x(t+T)= a_0+\sum_{n=1}^{k}[a_n \cos \omega_0 n(t+T) + b_n \sin \omega_0 n(t+T)]$

$= a_0+\sum_{n=1}^{k}[a_n \cos (\omega_0 nt + 2\pi n) + b_n \sin (\omega_0 nt + 2\pi n)]$

$= a_0+\sum_{n=1}^{k}[a_n \cos \omega_0 nt + b_n \sin \omega_0 nt]$

$=x(t)$

This proves that the signal $x(t)$, which is a summation of sine and cosine functions of frequencies $0$, $\omega_0$, $2\omega_0$, $\ldots$, $k\omega_0$, is a periodic signal with period $T$. By changing $a_n$s and $b_n$s, we can construct any periodic signal with period $T$. If $k \to \infty$ in the expression for $x(t)$, we obtain the Fourier series representation of any periodic signal $x(t)$. That is, any periodic signal can be represented as an infinite sum of sine and cosine functions which themselves are periodic signals of angular frequencies $0$, $\omega_0$, $2\omega_0$, $\ldots$, $k\omega_0$. This set of harmonically related sine and cosine functions, i.e. $\sin \omega_0 nt$ and $\cos \omega_0 nt$, $n=0,1,\ldots$ Forms a complete orthogonal set over the interval $t_0$ to $t_0+T$ where $T=2\pi/\omega_0$

The infinite series of sine and cosine terms of frequencies $0$, $\omega_0$, $2\omega_0$, $\ldots$, $k\omega_0$ is known as trigonometric form of Fourier series and can be written as:
\[
x(t) = \sum_{n=0}^{\infty} \left[ a_n \cos \omega_0 nt + b_n \sin \omega_0 nt \right]
\]
or
\[
x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \omega_0 nt + b_n \sin \omega_0 nt \right]
\]
where \(a_n\) and \(b_n\) are constants; the coefficient \(a_0\) is called the dc component; \(a_1 \cos \omega_0 t + b_1 \sin \omega_0 t\) the first harmonic, \(a_2 \cos 2\omega_0 t + b_2 \sin 2\omega_0 t\) the second harmonic and \(a_n \cos \omega_0 nt + b_n \sin \omega_0 nt\) the \(n\)th harmonic. The constant \(b_0 = 0\) because \(\sin \omega_0 nt = 0\) for \(n=0\).

**Evaluation of Fourier coefficients of the trigonometric Fourier series**

The constants \(a_0, a_1, a_2, \ldots, a_k\) and \(b_0, b_1, b_2, \ldots, b_k\) are called Fourier coefficients. To evaluate \(a_0\), we shall integrate both sides of the equation for \(x(t)\) over one period \(t_0\) to \(t_0+T\) at an arbitrary time \(t_0\) . Thus,

\[
\int_{t_0}^{t_0+T} x(t) \, dt = a_0 \int_{t_0}^{t_0+T} \left[ \sum_{n=1}^{\infty} \left[ a_n \cos \omega_0 nt + b_n \sin \omega_0 nt \right] \right] \, dt
\]

\[
= a_0 T + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos \omega_0 nt \, dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin \omega_0 nt \, dt
\]

We know that \(\int_{t_0}^{t_0+T} \cos \omega_0 nt \, dt = 0\) and \(\int_{t_0}^{t_0+T} \sin \omega_0 nt \, dt = 0\), since the net areas of sinusoids over complete periods are zero for any nonzero integer \(n\) and any time \(t_0\). Hence, each of the integrals in the above summation is zero.

Thus, we obtain

\[
\int_{t_0}^{t_0+T} x(t) \, dt = a_0 T \quad \text{or} \quad a_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) \, dt
\]

To evaluate \(a_n\) and \(b_n\), we can use the following results:

\[
\int_{t_0}^{t_0+T} \cos \omega_0 nt \cos \omega_0 mt \, dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n \neq 0 \end{cases}
\]

\[
\int_{t_0}^{t_0+T} \sin \omega_0 nt \sin \omega_0 mt \, dt = \begin{cases} 0 & \text{for } m \neq n \\ \frac{T}{2} & \text{for } m = n \neq 0 \end{cases}
\]

\[
\int_{t_0}^{t_0+T} \sin \omega_0 nt \cos \omega_0 mt \, dt = 0 \quad \text{for all } m \text{ and } n.
\]

To find Fourier coefficients \(a_n\), multiply the equation for \(x(t)\) by \(\cos \omega_0 mt\) and integrate over one period. That is,

\[
\int_{t_0}^{t_0+T} x(t) \cos \omega_0 mt \, dt = a_0 \int_{t_0}^{t_0+T} \cos \omega_0 mt \, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos \omega_0 nt \cos \omega_0 mt \, dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin \omega_0 nt \cos \omega_0 mt \, dt
\]

The first and third integrals in the above equation are equal to zero and the second integral is equal to \(T/2\) when \(m = n\). Therefore,

\[
\int_{t_0}^{t_0+T} x(t) \cos \omega_0 mt \, dt = a_m \frac{T}{2}
\]

\[
a_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \cos \omega_0 mt \, dt
\]

To find \(b_n\), multiply both sides of equation for \(x(t)\) by \(\sin \omega_0 mt\) and integrate over one period. Then

\[
\int_{t_0}^{t_0+T} x(t) \sin \omega_0 mt \, dt = a_0 \int_{t_0}^{t_0+T} \sin \omega_0 mt \, dt + \sum_{n=1}^{\infty} a_n \int_{t_0}^{t_0+T} \cos \omega_0 nt \sin \omega_0 mt \, dt + \sum_{n=1}^{\infty} b_n \int_{t_0}^{t_0+T} \sin \omega_0 nt \sin \omega_0 mt \, dt
\]

The first and second integrals in the above equation are zero, and the third integral is equal to \(T/2\) when \(m = n\). Thus, we have

\[
\int_{t_0}^{t_0+T} x(t) \sin \omega_0 mt \, dt = b_m \frac{T}{2}
\]
\[ b_m = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin \omega_m t \, dt \quad \text{or} \quad b_n = \frac{2}{T} \int_{t_0}^{t_0+T} x(t) \sin \omega_n t \, dt \]
a_0, a_n, and b_n are called trigonometric Fourier series coefficients.

A periodic signal has the same Fourier series for the entire interval \(-\infty\) to \(\infty\) as for the interval \(t_0\) to \(t_0+T\), since the same function repeats after every \(T\) seconds. The Fourier series expansion of a periodic function is unique irrespective of the location of \(t_0\) of the signal.

### Cosine representation (Alternate form of the Trigonometric Representation):

The trigonometric Fourier series of \(x(t)\) contains sine and cosine terms of the same frequency.

\[
x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \omega_n t + b_n \sin \omega_n t \right]
\]

or

\[
x(t) = a_0 + \sum_{n=1}^{\infty} \sqrt{a_n^2 + b_n^2} \left( \frac{a_n}{\sqrt{a_n^2 + b_n^2}} \cos \omega_n t + \frac{b_n}{\sqrt{a_n^2 + b_n^2}} \sin \omega_n t \right)
\]

Substituting the values \(A_0 = a_0\), \(A_n = \sqrt{a_n^2 + b_n^2}\), \(\cos \theta_n = \frac{a_n}{\sqrt{a_n^2 + b_n^2}}\), and \(\sin \theta_n = -\frac{b_n}{\sqrt{a_n^2 + b_n^2}}\)

i.e. \(\theta_n = -\tan^{-1} \frac{b_n}{a_n}\)

we have

\[
x(t) = a_0 + \sum_{n=1}^{\infty} A_n \left[ \cos \theta_n \cos \omega_n t - \sin \theta_n \sin \omega_n t \right]
\]

\[
x(t) = a_0 + \sum_{n=1}^{\infty} A_n \left[ \cos (\omega_n t + \theta_n) \right]
\]

This is the cosine representation of \(x(t)\) which contains sinusoids of frequencies \(\omega_0, 2\omega_0, \ldots\).

The term \(A_0\) is called the dc component, and the term \(A_n \left[ \cos (\omega_n t + \theta_n) \right]\) is called the \(n\)th harmonic component of \(x(t)\). The first harmonic component \(A_1 \left[ \cos (\omega_0 t + \theta_1) \right]\) is commonly called the fundamental component as it has the same fundamental period as \(x(t)\).

The number \(A_n\) represents amplitude coefficients or harmonic amplitudes or spectral amplitudes of the Fourier series and the number \(\theta_n\) represents the phase coefficients or phase angles or spectral phase of the Fourier series.

The cosine form is also called the Harmonic form Fourier series or Polar form Fourier series.

### Wave Symmetry:

If the periodic signal \(x(t)\) has some type of symmetry, then some of the trigonometric Fourier series coefficients may become zero and calculation of the coefficients becomes simple.

There are following four types of symmetry a function \(x(t)\) can have:

1. Even symmetry
2. Odd symmetry
3. Half wave symmetry
4. Quarter wave symmetry

If \(x(t)\) has even symmetry (also called mirror symmetry), then \(b_n = 0\) and only \(a_0\) and \(a_n\) are to be evaluated.

On the other hand, if \(x(t)\) has odd symmetry (also called rotation symmetry), then \(a_0 = 0\) and \(a_n = 0\). Only \(b_n\) is to be evaluated.

If \(x(t)\) has half wave symmetry, then \(a_0 = 0\) and only odd harmonics exists.

If \(x(t)\) has quarter wave symmetry, then \(a_0 = 0\) and either only \(a_n\) or only \(b_n\) exists, then too only for odd values of \(n\).

A function \(x(t)\) is said to be even if \(x(t) = x(-t)\)

A function \(x(t)\) is said to be odd if \(x(t) = -x(-t)\)

We know that any signal \(x(t)\) can be split into even and odd function. That is,

\[ x(t) = x_e(t) + x_o(t) \]

so the even and odd parts of the signal \(x(t)\) can be obtained from the following relations:
\[ x_e(t) = \frac{1}{2} [x(t) + x(-t)] \]
\[ x_o(t) = \frac{1}{2} [x(t) - x(-t)] \]

These relations can be used to find the Fourier coefficients. To have a convenient form, we choose the interval of integration from \(-\frac{T}{2}\) to \(\frac{T}{2}\) instead of from \(t_0\) to \(t_0 + T\). Also we know that

**Odd function × Odd function = Even function**

**Even function × Even function = Even function**

**Odd function × Even function = Odd function**

For any even function \(x_e(t)\),
\[ a_n = 2 \int_{\frac{T}{2}}^{\frac{T}{2}} x_e(t) \cos \omega_0 nt \, dt \]
and for any even function \(x_o(t)\),
\[ a_o = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos \omega_0 nt \, dt \]
and \(b_n = 0\) because \(x_e(t) \sin \omega_0 nt\) is an odd function.

Thus, the Fourier series expansion of an even periodic function contains only cosine terms and a constant.

When even or mirror symmetry exists, the trigonometric Fourier series coefficients are
\[ a_o = \frac{2}{T} \int_0^T x(t) \, dt \]
\[ a_n = \frac{4}{T} \int_0^T x(t) \cos \omega_0 nt \, dt \]
and \(b_n = 0\).

If the average value is 0, the coefficients \(a_o\) will be absent, i.e. \(a_o = 0\). The even function shows the mirror symmetry with respect to the vertical axis. Thus, the waveform having even symmetry contains only cosine terms and \(a_o\) may be present.

**1. Even or mirror symmetry**

A function \(x(t)\) is said to have even or mirror symmetry, if
\[ x(t) = x(-t) \]

The waveforms of some even function are shown in figure below. The sum of two or more even functions is always even. If a constant is added, the even nature of the function still persists. These types of functions are always symmetrical with respect to the vertical axis. The product of two even functions is always even.

If \(x(t)\) is an even function, then \(x_o(t) = 0\) and \(x(t) = x_e(t)\). Therefore,
\[ a_n = 2 \int_{-\frac{T}{2}}^{\frac{T}{2}} x_e(t) \cos \omega_0 nt \, dt = a_n = \frac{4}{T} \int_0^T x_e(t) \cos \omega_0 nt \, dt \]
and
\[ a_o = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x(t) \cos \omega_0 nt \, dt = \frac{1}{T} \int_{-\frac{T}{2}}^{\frac{T}{2}} x_e(t) \, dt = \frac{2}{T} \int_0^T x(t) \, dt \]
and
\[ b_n = 0 \]

because \(x_e(t) \sin \omega_0 nt\) is an odd function.

Thus, the Fourier series expansion of an even periodic function contains only cosine terms and a constant.
2. **Odd or Rotation symmetry**

A function \( x(t) \) is said to have odd or rotation symmetry, if

\[
x(t) = -x(-t)
\]

The waveforms of some odd functions are shown in the below figure. The sum of two or more odd functions is always odd. If a constant is added, the odd nature of the function is removed. These types of functions are always anti-symmetrical with respect to the vertical axis. The product of two odd functions is odd.

If \( x(t) \) is an odd function, then \( x_e(t) = 0 \) and \( x(t) = x_o(t) \)

\[
\therefore a_o = \frac{1}{T} \int_{-T/2}^{T/2} x(t) \, dt = \frac{1}{T} \int_{-T/2}^{T/2} x_o(t) \, dt = 0
\]

and

\[
a_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \cos \omega_0 nt \, dt = \frac{2}{T} \int_{-T/2}^{T/2} x_o(t) \cos \omega_0 nt \, dt = 0
\]

because \( x_o(t) \cos \omega_0 nt \) is an odd function.

\[b_n = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \sin \omega_0 nt \, dt = \frac{2}{T} \int_{-T/2}^{T/2} x_o(t) \sin \omega_0 nt \, dt = \frac{4}{T} \int_{0}^{T/2} x(t) \sin \omega_0 nt \, dt\]

Thus, the Fourier series expansion of an odd periodic function contains only sine terms. When odd or rotation symmetry exists, the trigonometric Fourier series coefficients are

\[a_o = 0, \quad a_n = 0, \quad \text{and} \quad b_n = \frac{4}{T} \int_{0}^{T/2} x(t) \sin \omega_0 nt \, dt\]
3. **Half wave symmetry**

A periodic signal $x(t)$ which satisfies the condition:

$$x(t) = -x(t \pm \frac{T}{2})$$

is said to have half wave symmetry. Figure shown below a function having half wave symmetry. This function is neither purely odd nor purely even. For such function, $a_n = 0$. The Fourier series expansion of such a type of periodic signal conditions odd harmonics only, that is $\omega_0, 3\omega_0, \ldots$ Etc.

As $x(t)$ contains only odd harmonic terms, when $n$ is even $a_n = b_n = 0$.

When $n$ is odd,

$$a_n = \frac{4}{T} \int_0^{T/2} x(t) \cos \omega_0 nt \, dt$$

and

$$b_n = \frac{4}{T} \int_0^{T/2} x(t) \sin \omega_0 nt \, dt$$

This can be proved as follows:

$$a_n = \frac{2}{T} \int_0^T x(t) \cos \omega_0 nt \, dt$$

$$= \frac{2}{T} \left( \int_0^{T/2} x(t) \cos \omega_0 nt \, dt + \int_{T/2}^T x(t) \cos \omega_0 nt \, dt \right)$$

Changing the variable from $t$ to $(t + T/2)$ in the second integral, we get

$$a_n = \frac{2}{T} \int_0^{T/2} x(t) \cos \omega_0 nt \, dt + \int_{T/2}^T x(t + T/2) \cos \omega_0 n(t + T/2) \, dt$$

using the property $x(t) = -x(t \pm \frac{T}{2})$, we get

$$a_n = \frac{2}{T} \left[ x(t) \cos \omega_0 nt - x(t) \cos \omega_0 nt \cos n\pi \right] \, dt$$

$$= \begin{cases} 
0 & \text{for even } n \\
\frac{4}{T} \int_0^{T/2} x(t) \cos \omega_0 nt \, dt & \text{for odd } n 
\end{cases}$$

Similarly, it can be shown that

$$b_n = \begin{cases} 
0 & \text{for even } n \\
\frac{4}{T} \int_0^{T/2} x(t) \sin \omega_0 nt \, dt & \text{for odd } n 
\end{cases}$$

4. **Quarter wave symmetry**:

A function $x(t)$ is said to have quarter wave symmetry, if

$$x(t) = x(-t) \text{ or } x(t) = -x(-t) \text{ and also } x(t) = -x(t \pm \frac{T}{2})$$

i.e. a function $x(t)$ which has either even symmetry or odd symmetry along with half wave symmetry is said to have quarter wave symmetry. Below Figure shows the examples of such functions.
Quarter wave symmetry
The following two cases are discussed below for the waveforms having quarter wave symmetry.

Case 1: \( x(t) = -x(-t) \) and \( x(t) = -x\left(t \pm \frac{T}{2}\right) \)

\[
\begin{align*}
    a_n &= 0, \\
    a_n &= 0, \\
    b_n &= \frac{8}{T} \int_{0}^{T/4} x(t) \sin \omega_0 nt \, dt \\
    \text{for odd } n
\end{align*}
\]

case 2: \( x(t) = x(-t) \) and also \( x(t) = -x\left(t \pm \frac{T}{2}\right) \)

\[
\begin{align*}
    a_n &= 0, \\
    a_n &= \frac{8}{T} \int_{0}^{T/4} x(t) \cos \omega_0 nt \, dt \\
    b_n &= 0 \\
    \text{for even } n
\end{align*}
\]

Exponential Fourier series:
The exponential Fourier series is the most widely used form of Fourier series. In this, the function \( x(t) \) is expressed as a weighted sum of the complex exponential functions. Although the trigonometric form and the cosine representation are the common forms of Fourier series, the complex exponential form is more general and usually more convenient and more compact. So, it is used almost exclusively, and it finds extensive application in communication theory.

The set of complex exponential functions
\[
\{e^{jn\omega_0 t}, n = 0, \pm 1, \pm 2, \ldots\}
\]
forms a closed orthogonal set over an interval \((t_0, t_0 + T)\) where \(T = (2\pi/\omega_0)\) for any value of \(t_0\), and therefore it can be used as a Fourier series. Using Euler’s identity, we can write

\[
A_n \cos(n\omega_0 t + \theta_n) = A_n \left[\frac{e^{i(n\omega_0 t + \theta_n)} + e^{-i(n\omega_0 t + \theta_n)}}{2}\right]
\]

Substituting this in the definition of the cosine Fourier representation, we obtain

\[
x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[ e^{i(n\omega_0 t + \theta_n)} + e^{-i(n\omega_0 t + \theta_n)} \right]
\]

\[
= A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} \left[ e^{in\omega_0 t} e^{i\theta_n} + e^{-in\omega_0 t} e^{-i\theta_n} \right]
\]

\[
= A_0 + \sum_{n=1}^{\infty} \left( \frac{A_n}{2} e^{i\theta_n} \right) e^{in\omega_0 t} + \sum_{n=1}^{\infty} \left( \frac{A_n}{2} e^{-i\theta_n} \right) e^{-in\omega_0 t}
\]

Letting \(n = -K\) in the second summation of the above equation, we have
\[ x(t) = A_0 + \sum_{n=1}^{\infty} \left( \frac{A_n}{2} e^{i\theta_n} \right) e^{jn\omega_0 t} + \sum_{k=-1}^{-\infty} \left( \frac{A_k}{2} e^{j\theta_k} \right) e^{jk\omega_0 t} \]

on comparing the above two equations for \( x(t) \), we get

\[ A_n = A_k; \quad (-\theta_n) = \theta_k \quad n > 0 \]
\[ \quad k < 0 \]

let us define

\[ C_0 = A_0; \quad C_n = \frac{A_n}{2} e^{i\theta_n}, \quad n > 0 \]

\[ \therefore \quad x(t) = A_0 + \sum_{n=1}^{\infty} \frac{A_n}{2} e^{i\theta_n} e^{jn\omega_0 t} + \sum_{n=-\infty}^{-1} \frac{A_n}{2} e^{i\theta_n} e^{jk\omega_0 t} \]

i.e. \[ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \]
This is known as exponential form of Fourier series. The above equation is called the synthesis equation.

So the exponential series from cosine series is:

\[ C_0 = A_0 \]
\[ C_n = \frac{A_n}{2} e^{i\theta_n} \]

**i. Determination of the coefficients of Exponential Fourier Series**

We have \[ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t}; \quad \omega_0 = \frac{2\pi}{T} \]

Multiplying both sides by \( e^{-jk\omega_0 t} \) and integrating over one period, we get

\[ \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = \int_{t_0}^{t_0+T} \left( \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} \right) e^{-jk\omega_0 t} dt \]
\[ = \sum_{n=-\infty}^{\infty} C_n \int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t} dt \]

We know that

\[ \int_{t_0}^{t_0+T} e^{jn\omega_0 t} e^{-jk\omega_0 t} dt = \begin{cases} 0 & \text{for } k \neq n \\ T & \text{for } k = n \end{cases} \]

\[ \therefore \quad \int_{t_0}^{t_0+T} x(t) e^{-jk\omega_0 t} dt = TC_k \]

or

\[ C_n = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) e^{-jn\omega_0 t} dt \]

where \( C_n \) are the Fourier coefficients of the exponential series. The above equation is called the analysis equation.

\[ C_0 = A_0 = \frac{1}{T} \int_{t_0}^{t_0+T} x(t) dt \]

The Fourier coefficients of \( x(t) \) have only a discrete spectrum because values of \( C_n \) exist only for discrete values of \( n \). As it represents a complex spectrum, it has both magnitude and phase spectra. The following points may be noted:

1. The magnitude line spectrum is always an even function of \( n \).
2. The phase line spectrum is always an odd function of \( n \).

**ii. Trigonometric Fourier Series from Exponential Fourier Series**

The complex exponential Fourier series is given by

\[ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = C_0 + \sum_{n=1}^{\infty} C_n e^{jn\omega_0 t} + \sum_{n=-1}^{\infty} C_n e^{-jn\omega_0 t} \]

\[ = C_0 + \sum_{n=1}^{\infty} \left( C_n e^{jn\omega_0 t} + C_n e^{-jn\omega_0 t} \right) \]

\[ = C_0 + \sum_{n=1}^{\infty} C_n (\cos n\omega_0 t - j\sin n\omega_0 t) + C_n (\cos n\omega_0 t + j\sin n\omega_0 t) \]

\[ x(t) = C_0 + \sum_{n=1}^{\infty} C_n \cos n\omega_0 t + j(C_n - C_{-n})\sin n\omega_0 t \]

Comparing this \( x(t) \) with the standard trigonometric Fourier series to trigonometric Fourier series as:

\[ a_0 = C_0 \]
\[ a_n = C_n + C_{-n} \]
Similarly, an exponential Fourier series can be derived from the trigonometric Fourier series

### Exponential Fourier series from Trigonometric Fourier series

From the exponential Fourier series, we know that

\[
C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega t} \, dt = \frac{1}{T} \int_0^T x(t) (\cos n\omega t - j\sin n\omega t) \, dt \\
= \frac{1}{2} \left( \int_0^{T/2} x(t) \cos n\omega t \, dt - j \int_0^{T/2} x(t) \sin n\omega t \, dt \right) = \frac{1}{2} [a_n - jb_n]
\]

\[
C_{-n} = \frac{1}{T} \int_0^T x(t) e^{jn\omega t} \, dt = \frac{1}{T} \int_0^T x(t) (\cos n\omega t + j\sin n\omega t) \, dt \\
= \frac{1}{2} \left( \int_0^{T/2} x(t) \cos n\omega t \, dt + j \int_0^{T/2} x(t) \sin n\omega t \, dt \right) = \frac{1}{2} [a_n + jb_n]
\]

\[
C_0 = \frac{1}{T} \int_0^T x(t) \, dt = a_0
\]

So, the formulae for conversion of trigonometric series to exponential series are:

\[
C_0 = a_0 \\
C_n = \frac{1}{2} (a_n - jb_n) \\
C_{-n} = \frac{1}{2} (a_n + jb_n)
\]

### Cosine Fourier Series from Exponential Fourier Series

We know that

\[
A_0 = a_0 = C_0 \\
A_n = \sqrt{a_n^2 + b_n^2} = \sqrt{(C_n + C_{-n})^2 + [j(C_n - C_{-n})]^2} \\
= \sqrt{(C_n^2 + C_{-n}^2 + 2C_n C_{-n}) - (C_n^2 + C_{-n}^2 - 2C_n C_{-n})} \\
= \sqrt{4C_n C_{-n}} \\
= 2 |C_n|
\]

\[
\therefore \quad A_0 = C_0 \\
A_n = 2 |C_n|
\]

### Solved Problems:

**Problem 1:** Find the Fourier series expansion of the half wave rectified sine wave shown in fig below

**Solution:**

The periodic waveform shown in fig with period \(2\pi\) is half of a sine wave with period \(2\pi\).

\[
x(t) = \begin{cases} 
A \sin \omega t = A \sin \frac{2\pi}{2\pi} t = A \sin t & 0 \leq t \leq \pi \\
0 & \pi \leq t \leq 2\pi 
\end{cases}
\]

Now the fundamental period \(T = 2\pi\)

Fundamental frequency \(\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1\)

Let \(t_0 = 0, t_0 + T = T = 2\pi\)
\[ a_0 = \frac{1}{T} \int_0^T x(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} x(t) \, dt = \frac{1}{2\pi} \int_0^{\pi} A \sin t \, dt = \frac{A}{\pi} \]
\[
\therefore a_0 = \frac{A}{\pi} \\
\]
\[ a_n = \frac{2}{T} \int_0^T x(t) \cos \omega_0 nt \, dt = \frac{2}{2\pi} \int_0^{2\pi} x(t) \cos nt \, dt = \frac{1}{\pi} \int_0^{\pi} A \sin t \cos nt \, dt \\
= -\frac{A}{2\pi} \left\{ \left\{ \frac{(-1)^{n+1}}{1+n} \right\} + \left\{ \frac{(-1)^n}{1-n} \right\} \right\}
\]
For odd \( n \), \( a_n = -\frac{A}{2\pi} \left\{ \left\{ \frac{1}{1+n} \right\} + \left\{ \frac{1}{1-n} \right\} \right\} = 0 \)
For even \( n \), \( a_n = -\frac{A}{2\pi} \left\{ \left\{ \frac{1}{1+n} \right\} + \left\{ \frac{1}{1-n} \right\} \right\} = -\frac{2A}{\pi(n^2-1)} \)
\[ b_n = \frac{2}{T} \int_0^T x(t) \sin \omega_0 nt \, dt = b_n = \frac{2}{2\pi} \int_0^{2\pi} x(t) \sin nt \, dt = \frac{1}{\pi} \int_0^{\pi} A \sin t \sin nt \, dt \\
= \left. \frac{A}{2\pi} \left[ \sin (n-1)t \right]_{-n+1} - \sin (n+1)t \right|_0^n \]
This is zero for all values of \( n \) except for \( n=1 \).

For \( n=1 \), \( b_1 = \frac{A}{2\pi} \)
Therefore, the trigonometric Fourier series is:
\[ x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \omega_0 nt + b_n \sin \omega_0 nt \right] \]
\[ = a_0 + b_1 \sin t + \sum_{n=1}^{\infty} a_n \cos nt \]
\[ = \frac{A}{\pi} + \frac{A}{2\pi} \sin t - \sum_{n=1}^{\infty} \frac{2A}{\pi(n^2-1)} \cos nt \]

**Problem 2:** Obtain the trigonometric Fourier series for the waveform shown in below fig

**Solution:**
The waveform shown in fig above is periodic with a period \( T=2\pi \)
Let \( t_0 = 0, \ t_0 + T = T = 2\pi \)
Fundamental frequency \( \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \)
The waveform is described by
\[ x(t) = \begin{cases} \left( \frac{A}{\pi} \right) t & 0 \leq t \leq \pi \\ 0 & \pi \leq t \leq 2\pi \end{cases} \]
\[ a_0 = \frac{1}{T} \int_0^T x(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} x(t) \, dt = \frac{1}{2\pi} \int_0^{\pi} \frac{A}{\pi} t \, dt = \frac{A}{2\pi^2} \left[ \frac{t^2}{2} \right]_0^\pi = \frac{A}{4} \]
\[ a_n = \frac{2}{T} \int_0^T x(t) \cos \omega_0 nt \, dt = \frac{2}{2\pi} \int_0^{2\pi} \frac{A}{\pi} t \cos nt \, dt = \frac{A}{\pi n^2} \left( \cos nt \pi - \cos 0 \right) \\
\therefore a_n = \begin{cases} -\frac{2A}{\pi n^2} & \text{for odd } n \\ 0 & \text{for even } n \end{cases} \]
\[ b_n = \frac{2}{T} \int_0^T x(t) \sin \omega_0 nt \, dt = \frac{2}{2\pi} \int_0^{2\pi} x(t) \sin nt \, dt = \frac{1}{\pi} \int_0^{\pi} \frac{A}{\pi} t \sin nt \, dt = \frac{A}{\pi n} (-1)^{n+1} \]
\[ b_n = \begin{cases} \left( \frac{A}{\pi n} \right) & \text{for odd } n \\ -\left( \frac{A}{\pi n} \right) & \text{for even } n \end{cases} \]

The trigonometric Fourier series is:

\[ x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \omega_0 nt + b_n \sin \omega_0 nt \right] \]

\[ = \frac{A}{4} - \frac{2A}{\pi^2} \lim_{N \to \infty} \sum_{n=1}^{N} \frac{\cos nt}{n^2} + \frac{A}{\pi} \sum_{n=1}^{N} \frac{(-1)^{n+1} \sin nt}{n} \]

\[ = \frac{A}{4} - \frac{2A}{\pi^2} \left[ \frac{\sin t - \frac{1}{2} \sin 2t + \frac{1}{3} \sin 3t - \frac{1}{4} \sin 4t + \cdots}{\pi} \right] \]

**Problem 3:** Find the trigonometric Fourier series for the waveform shown in below fig.

Solution:
The waveform shown in figure is given by

\[ x(t) = \frac{A}{2\pi} t, \text{ for } 0 \leq t \leq 2\pi \]

Let \( t_0 = 0, t_0 + T = T = 2\pi \)

and \( \text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = 2\pi = 1 \)

The coefficients are evaluated as follows:

\[ a_0 = \frac{1}{T} \int_0^T x(t) \, dt = \frac{1}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} t \, dt = -\frac{A}{(2\pi)^2} \left[ \frac{t^2}{2} \right]_0^T = \frac{A}{2} \]

\[ a_n = \frac{2}{T} \int_0^T x(t) \cos \omega_0 nt \, dt = \frac{2}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} t \cos nt \, dt \]

\[ = \frac{2A}{(2\pi)^2} \left[ 0 - 0 + \frac{1}{n} \frac{(1-1)}{n} \right] = 0 \]

\[ b_n = \frac{2}{T} \int_0^T x(t) \sin \omega_0 nt \, dt = \frac{2}{2\pi} \int_0^{2\pi} \frac{A}{2\pi} t \sin nt \, dt = -\frac{A}{\pi n} \]

\[ a_0 = \frac{A}{2} ; a_n = 0 ; b_n = -\frac{A}{\pi n} \]

are the trigonometric Fourier series coefficients.

\[ x(t) = \frac{A}{2} + \sum_{n=1}^{\infty} -\frac{A}{\pi n} \sin nt = -\frac{A}{\pi} \sum_{n=1}^{\infty} \frac{\sin nt}{n} \]

\[ 0 \leq t \leq 2\pi \]

is the trigonometric Fourier series coefficients.

**Problem 4:** Find the Fourier series expansion for the waveform shown in figure below:
Solution:
The given waveform of figure is periodic with period $T = 2\pi$. For the computational convenience, choose one cycle of the waveform from $-\pi$ to $\pi$.

Let $t_0 = -\pi$, $t_0 + T = \pi$

The fundamental frequency is $\omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1$

The waveform is described by

$x(t) = \begin{cases} \frac{-A}{\pi} t & \text{for } -\pi \leq t \leq 0 \\ \frac{A}{\pi} t & \text{for } 0 \leq t \leq \pi \end{cases}$

The given waveform has even symmetry because $x(t) = x(-t)$.

\[ a_0 = \frac{2}{T} \int_{0}^{T/2} x(t) \, dt, \quad a_n = \frac{4}{T} \int_{0}^{T/2} x(t) \cos \omega_0 nt \, dt \quad \text{and} \quad b_n = 0 \]

Now,
\[ a_0 = \frac{2}{T} \int_{0}^{T/2} x(t) \, dt = \frac{2}{T} \int_{0}^{T/2} \frac{A}{\pi} t \, dt = \frac{A}{(\pi)^2} \left[ \frac{t^2}{2} \right]_{0}^{\pi} = \frac{A}{2} \]

By observation also we can say that the average value $a_0 = \frac{A}{2}$

\[ a_n = \frac{4}{2\pi} \int_{0}^{T/2} x(t) \cos \omega_0 nt \, dt = \frac{4A}{\pi^2 n^2} \int_{0}^{T/2} \cos nt \, dt = \frac{2A}{\pi^2 n^2} (\cos n\pi - \cos 0) = \frac{2A}{\pi^2 n^2} [(-1)^n - 1] \]

\[ a_n = \begin{cases} -\left( \frac{4A}{\pi^2 n^2} \right) & \text{for odd } n \\ 0 & \text{for even } n \end{cases} \]

The trigonometric Fourier series is:

\[ x(t) = a_0 + \sum_{n=1}^{\infty} [a_n \cos \omega_0 nt + b_n \sin \omega_0 nt] \]

\[ = \frac{A}{2} + \sum_{n=\text{odd}}^{\infty} -\left( \frac{4A}{\pi^2 n^2} \right) \cos nt \]

\[ = \frac{A}{2} - \left( \frac{4A}{\pi^2 n^2} \right) \sum_{n=\text{odd}}^{\infty} \left( \frac{\cos nt}{n^2} \right) \]

Problem 5: Obtain the Fourier components of the periodic rectangular waveform shown in below figure.
Solution: The waveform shown in figure is a periodic waveform with period = T and can be expressed as:

\[
x(t) = \begin{cases} 
0 & \text{for } -\frac{T}{2} \leq t \leq -\frac{T}{4} \\
A & \text{for } -\frac{T}{4} \leq t \leq \frac{T}{4} \\
0 & \text{for } \frac{T}{4} \leq t \leq \frac{T}{2}
\end{cases}
\]

Let 
\[ t_0 = -\frac{T}{2}, t_0 + T = \frac{T}{2} \]
and Fundamental frequency \( \omega_0 = \frac{2\pi}{T} \)

The given function has even symmetry because \( x(t) = x(-t) \).

\[
\therefore a_0 = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \, dt, \quad a_n = \frac{4}{T} \int_{T/4}^{T/4} x(t) \cos \omega_0 nt \, dt \quad \text{and} \quad b_n = 0
\]

Now,
\[
a_0 = \frac{2}{T} \int_{-T/2}^{T/2} x(t) \, dt = 2A \int_{-T/4}^{T/4} \frac{T/2}{T} A \, dt = A \]
\[
a_n = \frac{2}{T} \int_{-T/4}^{T/4} x(t) \cos \omega_0 nt \, dt = 2A \int_{-T/4}^{T/4} \frac{T/2}{T} x(t) \cos \omega_0 nt \, dt
\]
\[
= \frac{4}{T} \int_{0}^{T/4} A \cos n\frac{2\pi}{T} t \, dt = \frac{2A}{n\pi} \sin \frac{n\pi}{2}
\]

\[
a_n = \begin{cases} 
\frac{2A}{n\pi} & \text{for } n = 1, 5, 9, ... \\
-\frac{2A}{n\pi} & \text{for } n = 3, 7, 11, ... \\
0 & \text{for even } n
\end{cases}
\]

The trigonometric Fourier series is:
\[
x(t) = a_0 + \sum_{n=1}^{\infty} \left[ a_n \cos \omega_0 nt + b_n \sin \omega_0 nt \right]
\]
\[
= \frac{A}{2} + \sum_{n=\text{odd}}^{\infty} \left( \frac{2A}{n\pi} \sin \left( \frac{n\pi}{2} \right) \cos \left( \frac{2n\pi}{T} \right) \right) t
\]

Problem 6: Obtain the exponential Fourier Series for the wave form shown in below figure.

Solution: The periodic waveform shown in fig with a period T= 2π can be expressed as:
\[ x(t) = \begin{cases} 
A & 0 \leq t \leq \pi \\
-A & \pi \leq t \leq 2\pi 
\end{cases} \]

Let 
\[ t_0 = 0, \; t_0 + T = 2\pi \]
and 
Fundamental frequency \( \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{2\pi} = 1 \)

Exponential Fourier series

\[ C_0 = \frac{1}{T} \int_0^T x(t) \, dt \]
\[ = \frac{1}{2\pi} \int_0^\pi A \, dt + \frac{1}{2\pi} \int_\pi^{2\pi} -A \, dt = 0 \]

\[ C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} \, dt \]
\[ = \frac{1}{2\pi} \int_0^\pi Ae^{-jnt} \, dt + \frac{1}{2\pi} \int_\pi^{2\pi} -Ae^{-jnt} \, dt \]
\[ = -\frac{A}{j2\pi n} [(-1)^n - 1] - [1 - (-1)^n] = -j \frac{A}{2n\pi} \]

\[ C_n = \begin{cases} 
\left(\frac{-2A}{\pi n}\right) & \text{for odd } n \\
0 & \text{for even } n 
\end{cases} \]

\[ \therefore x(t) = C_0 + \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} -j \frac{A}{2n\pi} e^{jnt} \]

**Problem 7:** Find the exponential Fourier series for the full wave rectified sine wave given in below figure.

![Figure](image)

**Solution:** The waveform shown in fig can be expressed over one period (0 to \( \pi \)) as:

\[ x(t) = A \sin \omega t \] where \( \omega = \frac{2\pi}{2\pi} = 1 \)

because it is part of a sine wave with period = 2\( \pi \)

\[ x(t) = A \sin \omega t \quad 0 \leq t \leq \pi \]

The full wave rectified sine wave is periodic with period \( T = \pi \)

Let 
\[ t_0 = 0, \; t_0 + T = 0 + \pi = \pi \]
and 
Fundamental frequency \( \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{\pi} = 2 \)

The exponential Fourier series is

\[ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{j2nt} \]
where \( C_n = \frac{1}{T} \int_0^T x(t)e^{-jn\omega_0 t} \, dt \)

\[
= \frac{1}{\pi} \int_0^\pi \sin t \, dt = -\frac{A}{\pi} \int_0^\pi \sin t \, dt
\]

\[
= A \frac{\sin(1-2n)\pi - e^0}{j(1-2n)} - \frac{\sin(j(1-2n)\pi - e^0)}{-j(1-2n)}
\]

\[C_n = \frac{2A}{\pi(1-4n^2)}
\]

\[
C_0 = \frac{1}{T} \int_0^T x(t) \, dt = \frac{A}{\pi} \left[ -\cos t \right]_0^1 = \frac{2A}{\pi}
\]

The exponential Fourier series is given by

\[
x(t) = \sum_{n=-\infty}^{\infty} \frac{2A}{\pi(1-4n^2)} e^{jn\omega_0 t} = \frac{2A}{\pi} + \frac{2A}{\pi} \sum_{n=-\infty}^{\infty} \left( e^{jn\omega_0 t} \right)
\]

**Problem 8:** Find the exponential Fourier series for the rectified sine wave shown in below figure

![Rectified Sine Wave](image)

**Solution:**

The waveform shown in figure is a part of a sine wave with period \( =2 \)

\[ x(t) = A \sin \omega_0 t \quad 0 \leq t \leq 1, \text{ where } \omega = \frac{2\pi}{2} = \pi \]

hence \[ x(t) = A \sin \pi t \quad \text{for} \quad 0 \leq t \leq 1 \]

The period of the rectified sine wave is \( T = 1 \)

Let \[ t_0 = 0, \quad t_0 + T = 1 \]

and \[ \text{Fundamental frequency } \omega_0 = \frac{2\pi}{T} = \frac{2\pi}{1} = 2\pi \]

The exponential Fourier series is

\[ x(t) = \sum_{n=-\infty}^{\infty} C_n e^{jn\omega_0 t} = \sum_{n=-\infty}^{\infty} C_n e^{jn2\pi t} \]

where

\[ C_n = \frac{1}{T} \int_0^T x(t)e^{-jn\omega_0 t} \, dt = \frac{1}{\pi} \int_0^1 \sin t \, dt = \frac{A}{\pi} \int_0^1 \sin t \, dt
\]

\[= A \frac{\sin(1-2n)\pi - e^0}{j(1-2n)} - \frac{\sin(j(1-2n)\pi - e^0)}{-j(1-2n)}
\]

\[C_0 = \frac{2A}{\pi(1-4n^2)}
\]

\[C_0 = \frac{1}{T} \int_0^T x(t) \, dt = \frac{A}{\pi} \left[ -\cos t \right]_0^1 = \frac{2A}{\pi}
\]

The exponential Fourier series is:
\[ x(t) = \frac{2A}{\pi} + \frac{2A}{\pi} \sum_{n=-\infty}^{\infty} \frac{e^{j2n\pi t}}{1-4n^2} \]

**Problem 9:** Derive the exponential Fourier series from the trigonometric Fourier series for the waveform shown in figure.

Also find the exponential Fourier series directly.

**Solution:** Derivation of exponential series from trigonometric series:

From Example 2, we got the trigonometric Fourier coefficients as:

- \( a_0 = \frac{A}{4} \)
- \( a_n = \frac{2A}{\pi^2 n^2} \) (for odd \( n \)) and \( b_n = \frac{A}{\pi n} (-1)^{n+1} \)

\[ C_0 = a_0 = \frac{A}{4} \]

\[ C_n = \frac{1}{2} (a_n - j b_n) = \frac{1}{2} \left[ -\frac{2A}{\pi^2 n^2} - j \frac{A}{\pi n} (-1)^{n+1} \right] \]

Direct derivation of exponential Fourier series:

\[ x(t) = C_0 + \sum_{n=\pm 1}^{\infty} C_n e^{j \omega_0 t} = C_0 + \sum_{n=\pm 1}^{\infty} C_n e^{j \omega_0 t} \]

\[ C_0 = \frac{1}{T} \int_{0}^{T} x(t) dt = \frac{1}{2\pi} \int_{0}^{2\pi} x(t) dt = \frac{1}{2\pi} \int_{0}^{\pi} \frac{A}{t} dt = \frac{A}{4} \]

\[ C_n = \frac{1}{T} \int_{0}^{T} x(t) e^{-j \omega_0 t} dt = \frac{1}{2\pi} \int_{0}^{2\pi} x(t) e^{-j \omega_0 t} dt = \frac{1}{2\pi} \int_{0}^{2\pi} \frac{A}{t} e^{-j \omega_0 t} dt \]

\[ = \frac{A}{2\pi^2} \left\{ -\frac{\pi}{jn} (-1)^n - \frac{1}{(jn)^2} [(-1)^n - 1] \right\} \]

\[ \therefore \quad x(t) = \frac{A}{4} + \sum_{n=\pm 1}^{\infty} \frac{A}{2\pi^2} \left\{ -\frac{\pi}{jn} (-1)^n - \frac{1}{(jn)^2} [(-1)^n - 1] \right\} e^{-j \omega_0 t} \]

**Problem 10:** Find the complex exponential Fourier series for the waveform shown in figure below

The waveform shown in figure is periodic with period = \( T \) and can be expressed as

\[ x(t) = \begin{cases} A & \text{for } 0 \leq t \leq \frac{T}{2} \\ 0 & \text{for } \frac{T}{2} \leq t \leq T \end{cases} \]

fundamental frequency \( \omega_0 = \frac{2\pi}{T} \)

\[ C_0 = \frac{1}{T} \int_{0}^{T} x(t) dt = \frac{1}{T} \int_{0}^{T/2} A dt = \frac{A}{2} \]
\[ C_n = \frac{1}{T} \int_0^T x(t) e^{-jn\omega_0 t} \, dt = \frac{1}{T} \int_0^{T/2} A e^{-jn\frac{2\pi}{T} t} \, dt \]
\[ = \frac{A}{jn2\pi} [1 - (-1)^n] \]
\[ C_n = \begin{cases} \left( \frac{A}{\sqrt{\pi n}} \right) & \text{for odd } n \\ 0 & \text{for even } n \end{cases} \]
\[ \therefore \quad x(t) = \frac{A}{2} + \sum_{n=odd}^{\infty} \left( \frac{A}{\sqrt{\pi n}} \right) e^{-jn\pi t} \]

Assignment Problems:

1. Find the trigonometric Fourier series for the periodic signal \( x(t) \) shown in figure below:

![Figure 1](image1.png)

2. Find the trigonometric Fourier series for the periodic signal \( x(t) \) shown in figure below:

![Figure 2](image2.png)

3. Compute the exponential Fourier series of the signal shown in figure below:

![Figure 3](image3.png)

4. Find the exponential series of the signal shown in the figure below:

![Figure 4](image4.png)
5. Find the trigonometric Fourier series of \( x(t) = t^2 \) over the interval \((-1, 1)\).

6. Find the exponential series of the triangular waveform shown in figure below:

![Triangular Waveform](image)

7. Find the exponential Fourier series coefficient of the signal \( x(t) = 2 + \cos \left( \frac{2\pi t}{3} \right) + 4 \sin \left( \frac{5\pi t}{3} \right) \)

8. Find the complex exponential Fourier series coefficient of the signal \( x(t) = 3\cos 4\omega_0 t \).

9. Find the complex exponential Fourier series coefficient of the signal \( x(t) = 2\cos 3\omega_0 t \).

10. Find the complex exponential Fourier series coefficient of the signal
    
    (a) \( x(t) = \sin^2 t \)  
    (b) \( x(t) = e^{2t} \) with \( T=2 \)

**Simulation:**

**Approximating with Fourier Sine Series**

A MATLAB code is used to plot the square wave function along with the Fourier sine series in order to compare the accuracy and error between the approximation and the actual function.

**Program:**

```matlab
clear all
N = input( 'type the total number of harmonics' );
t = 0 : 0.001 : 1;
y = square( 2 * pi * t );
plot( t, y, 'r', 'linewidth', 2 )
axis( [ 0 1 -1.5 1.5 ] )
hold;
sq = zeros( size(t) );
for n = 1 : 2 : N
    sq = sq + (4 / (pi * n) * sin( 2 * pi * n * t));
end;
plot( t, sq )
grid;
xlabel( 'time' );
ylabel( 'amplitude' );
title('Synthesized Square Wave Using Fourier Series - Gibbs Phenomenon');
```

**Output for \( N = 5 \) Harmonics:**
Output for $N = 25$ Harmonics:

References:


